To Another Definiton From Historical Definition Of The Gamma Function

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Definition (Euler, 1730):

Let $x > 0$

$$\Gamma(x) = \int_0^1 (-logt)^{x-1}\, dt$$

where $logx$ is lnx.

Theorem:

For $x > 0$

$$\Gamma(x) = 2\int_0^\infty t^{2x-1}\, e^{-t^2} dt\, .$$

Proof:

Consider for $x > 0$

$$\Gamma(x) = \int_0^1 (-logt)^{x-1}\, dt.$$

Let $u^2 = -logt$, then

$$t = e^{-u^2} \qquad dt = -2ue^{-u^2} du$$
$$t = 0 \to u = \infty \qquad t = 1 \to u = 0$$

$$\Gamma(x) = \int_\infty^0 (u^2)^{x-1} \left(-2ue^{-u^2} du\right)$$

$$\Gamma(x) = 2\int_0^\infty u^{2x-1}\, e^{-u^2} du$$

$$\Gamma(x) = 2\int_0^\infty t^{2x-1}\, e^{-t^2} dt$$

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