Expresssion of factorial and gamma function by Euler's product

Abdullah Aydemir, 24.7.2023

İstanbul, Türkiye

https://www.abdullahaydemir.com.tr/

Abstract:

In our study, we first expressed and proved a lemma that we will use in the proof of theorem. Then we state and prove the theorem for factorial. Finally, we arrived at the definition of the gamma function by the Euler's product.

Lemma:

For $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{n!\ (n+1)^m}{(n+m)!} = 1\ .$$

Proof:

$$\lim_{n\to\infty} \frac{n!\ (n+1)^m}{(n+m)!} = \lim_{n\to\infty} \frac{n!\ (n+1)^m}{n!\,(n+1)(n+2)\cdots(n+m)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^m}{(n+1)(n+2)\cdots(n+m)}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)\cdots(n+1)}{(n+1)(n+2)\cdots(n+m)}$$

$$= \lim_{n\to\infty} \frac{n^m\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{1}{n}\right)}{n^m\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{m}{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{m}{n}\right)}$$

$$= 1\ .$$

Therefore, for $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{n!\ (n+1)^m}{(n+m)!} = 1\ .$$

Theorem: For $z \in \mathbb{N}$,

$$z! = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1}.$$

Proof:

From the lemma,

$$\lim_{n\to\infty} \frac{n!\,(n+1)^{z}}{(n+z)!} = 1.$$

By multiply with $z!$ both sides of this equality, we obtain

$$\lim_{n\to\infty} \frac{n!\,(n+1)^{z} z!}{(n+z)!} = z! \rightarrow z! = \lim_{n\to\infty} \frac{n!\,(n+1)^{z} z!}{z!\,(z+1)(z+2)\ldots(n+z)}$$

$$z! = \lim_{n\to\infty} \frac{n!\,(n+1)^{z}}{(1+z)(2+z)\ldots(n+z)} \rightarrow z! = \lim_{n\to\infty} \frac{(1\cdot 2\cdot \cdots \cdot n)\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\right)^{z}}{(1+z)(2+z)\cdots(n+z)}$$

$$z! = \lim_{n\to\infty} \frac{\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\right)^{z}}{\left(\frac{1+z}{1}\right)\left(\frac{2+z}{2}\right)\cdots\left(\frac{n+z}{n}\right)} \rightarrow z! = \lim_{n\to\infty} \frac{\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}\right)^{z}}{\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{n}\right)}$$

$$z! = \lim_{n\to\infty} \prod_{k=1}^{n} \frac{\left(1+\frac{1}{k}\right)^{z}}{\left(1+\frac{z}{k}\right)} \rightarrow z! = \prod_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^{z}}{\left(1+\frac{z}{n}\right)}$$

$$z! = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1} \quad \blacksquare$$

Corollary:

$$z! = \Gamma(z+1) = z\,\Gamma(z) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1}$$

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{z} \left(1 + \frac{z}{n}\right)^{-1}$$

where z can be taken as $z > 0$. The last expression is the definition of the gamma function by the Euler's product.