An important product: Wallis' s product

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Theorem:

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \frac{\pi}{2}.$$

Proof:

We use the Weierstrass Factorization of Sine for proof.

Since $Sinx = 0 \rightarrow x = 0, \mp\pi, \mp 2\pi, \mp 3\pi, \mp 4\pi, \cdots$, we have

$$Sinx = Ax\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots$$

$$\frac{Sinx}{x} = A\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots, A = 1 \; from \; \lim_{x \to 0} \frac{Sinx}{x} = 1$$

$$\frac{Sinx}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$$

We substitute $x = \frac{\pi}{2}$

$$\frac{Sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2}{\pi} = \prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) \rightarrow \frac{2}{\pi} = \prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right) \rightarrow \frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2-1}\right)$$

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2} \quad \blacksquare$$