Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

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Solution:

$$\Gamma(x) = \int_0^{\infty} t^{x-1}\, e^{-t} dt$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} u^{-\frac{1}{2}}\, e^{-u} du = \int_0^{\infty} v^{-\frac{1}{2}}\, e^{-v} dv$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \left(\int_0^{\infty} u^{-\frac{1}{2}}\, e^{-u} du\right)\left(\int_0^{\infty} v^{-\frac{1}{2}}\, e^{-v} dv\right)$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \int_0^{\infty}\int_0^{\infty} u^{-\frac{1}{2}}\, v^{-\frac{1}{2}} e^{-(u+v)} du dv$$

$$\begin{bmatrix} u = x^2 & v = y^2 \\ du = 2xdx & dv = 2ydy \end{bmatrix}$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \int_0^{\infty}\int_0^{\infty} 4e^{-(x^2+y^2)}\, dxdy$$

$$\begin{bmatrix} x = rCos\theta & r^2 = x^2 + y^2 \\ y = rSin\theta & dxdy = rdrd\theta \end{bmatrix}$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \int_0^{\frac{\pi}{2}}\int_0^{\infty} 4re^{-r^2}\, drd\theta = \left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^{\infty} 4re^{-r^2}\, dr\right) = \left(\frac{\pi}{2}\right)\left(-2e^{-r^2}\Big]_0^{\infty}\right) = \left(\frac{\pi}{2}\right)(2) = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$