A property of the Gamma function-By **Abdullah Aydemir,** 14.12.2022

For $n \in \mathbb{Z}^+$,

$$\Gamma\left(-n+\frac{1}{2}\right) = (-1)^n \frac{2^n}{1.3.5 \ldots (2n-1)} \sqrt{\pi}.$$

Proof:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(x) = \frac{\Gamma(x+1)}{x}$$

$$x = -1 + \frac{1}{2} = -\frac{1}{2} \Longrightarrow \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$x = -2 + \frac{1}{2} = -\frac{3}{2} \Longrightarrow \Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = \left(-\frac{2}{3}\right)(-2\sqrt{\pi})$$
$$= (-1)^2 \frac{2^2}{1.3}\sqrt{\pi}$$

$$x = -3 + \frac{1}{2} = -\frac{5}{2} \Longrightarrow \Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{5}\Gamma\left(-\frac{3}{2}\right) = \left(-\frac{2}{5}\right)\left((-1)^2\frac{2^2}{1.3}\sqrt{\pi}\right)$$
$$= (-1)^3 \frac{2^3}{1.3.5}\sqrt{\pi}$$

...

$$x = -n + \frac{1}{2} = \frac{-2n+1}{2} \Longrightarrow \Gamma\left(\frac{-2n+1}{2}\right) = \frac{\Gamma\left(\frac{-2n+1}{2}+1\right)}{\frac{-2n+1}{2}} = \frac{2}{-2n+1}\Gamma\left(\frac{-2n+3}{2}\right)$$
$$= \left(-\frac{2}{2n-1}\right)\left((-1)^{n-1}\frac{2^{n-1}}{1.3.5 \ldots (2n-3)}\sqrt{\pi}\right) = (-1)^n \frac{2^n}{1.3.5 \ldots (2n-1)}\sqrt{\pi}$$

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