An important property of the gamma function: The difference/functional/recurrence equation

Abdullah Aydemir, 20.1.2023

Theorem:

$$\Gamma(z+1) = z.\Gamma(z) \text{ for all } Re(z) > 0.$$

Proof:

Method 1:

Consider the integral definition of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}\, e^{-t} dt, \qquad Re(z) > 0.$$

Hence, we write

$$\Gamma(z+1) = \int_0^\infty t^{z}\, e^{-t} dt.$$

By integrating by parts we have

$$\begin{bmatrix} u = t^z & dv = e^{-t}dt \\ du = zt^{z-1}dt & v = -e^{-t} \end{bmatrix}$$

$$\int u\, dv = uv - \int v\, du$$

$$\Gamma(z+1) = -t^z e^{-t}\Big|_0^\infty - \int_0^\infty -e^{-t}\, zt^{z-1} dt$$

$$\Gamma(z+1) = 0 + z\int_0^\infty t^{z-1}\, e^{-t} dt$$

$$\Gamma(z+1) = z.\Gamma(z)$$

⊡

Method 2:

For $n \in \mathbb{N}$, we are knowing that

$$\Gamma(n) = (n-1)!\,.$$

Hence, we can write

$$\Gamma(n+1) = n! = n.(n-1)! = n\,\Gamma(n)\,.$$

For $x \in \mathbb{R} - (\mathbb{Z}^- \cup \{0\})$, we can extend to reals:

$$\Gamma(x+1) = x\,\Gamma(x)\,,$$

Also, for $z \in \mathbb{C}$ and $Re(z) > 0$, we can extend to complexes:

$$\Gamma(z+1) = z\,\Gamma(z)$$

$$\boxdot$$

Method 3:

$$\Gamma(z+1) = z.\Gamma(z)\,, \qquad Re(z) > 0$$

$$\Gamma(z+1) - z.\Gamma(z) = \int_0^\infty t^z\, e^{-t} dt - z\int_0^\infty t^{z-1}\, e^{-t} dt$$

$$= \int_0^\infty (t^z - zt^{z-1})\, e^{-t} dt$$

$$= \int_0^\infty d(-t^z e^{-t})$$

$$= -t^z e^{-t}\Big|_0^\infty$$

$$= 0$$

$$\boxdot$$

Method 4:

We are using the limit definition of the gamma function.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\ n^z}{z(z+1)(z+2)\cdots(z+n)}\ , \qquad z \neq 0, -1, -2, \cdots$$

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!\ n^{z+1}}{(z+1)(z+2)\cdots(z+n)(z+1+n)}$$

$$= \lim_{n\to\infty} \frac{n!\ n^z\ n}{(z+1)(z+2)\cdots(z+n)(z+1+n)}$$

$$= \lim_{n\to\infty} \frac{n!\ n^z}{(z+1)(z+2)\cdots(z+n)} \cdot \lim_{n\to\infty} \frac{n}{z+1+n}$$

Since

$$\lim_{n\to\infty} \frac{n}{z+1+n} = 1$$

we have

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{n!\ n^z}{(z+1)(z+2)\cdots(z+n)}$$

$$= \lim_{n\to\infty} \frac{n!\ n^z}{(z+1)(z+2)\cdots(z+n)} \cdot \frac{z}{z}$$

$$= z \lim_{n\to\infty} \frac{n!\ n^z}{z(z+1)(z+2)\cdots(z+n)}$$

$$= z\,\Gamma(z)\ .$$

$\boxdot$

Method 5:

We consider the Euler's product for $\Gamma(z)$:

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}.$$

Hence, we can write that

$$\Gamma(z+1) = \frac{1}{z+1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}.$$

So,

$$\frac{\Gamma(z+1)}{\Gamma(z)} = \frac{\frac{1}{z+1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{z+1}{n}\right)^{-1}}{\frac{1}{z}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}}$$

$$= \frac{z}{z+1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)\left(\frac{n+z+1}{n+z}\right)^{-1}$$

$$= \frac{z}{z+1}\prod_{n=1}^{\infty}\left(\frac{n+1}{n}\right)\left(\frac{n+z}{n+z+1}\right)$$

$$= \frac{z}{z+1}\lim_{n\to\infty}\prod_{k=1}^{n}\left(\frac{k+1}{k}\cdot\frac{k+z}{k+z+1}\right)$$

$$= \frac{z}{z+1}\lim_{n\to\infty}\left(\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n}{n-1}\cdot\frac{n+1}{n}\right)\left(\frac{1+z}{2+z}\cdot\frac{2+z}{3+z}\cdots\frac{n-1+z}{n+z}\cdot\frac{n+z}{n+z+1}\right)$$

$$= \frac{z}{z+1}\lim_{n\to\infty}(n+1)\left(\frac{1+z}{n+z+1}\right)$$

$$= z\lim_{n\to\infty}\left(\frac{n+1}{n+z+1}\right)$$

$$= z.1$$

$$= z$$

Therefore, we deduce that

$$\frac{\Gamma(z+1)}{\Gamma(z)} = z \;\Rightarrow\; \Gamma(z+1) = z\,\Gamma(z),$$

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Method 6:

We consider the Weierstrass's product for $\Gamma(z)$:

$$\frac{1}{z\,\Gamma(z)} = e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

where γ is the Euler-Mascheroni constant. Hence,

$$\frac{1}{(z+1)\,\Gamma(z+1)} = e^{\gamma(z+1)} \prod_{n=1}^{\infty} \left(1 + \frac{z+1}{n}\right) e^{-\frac{z+1}{n}}.$$

So,

$$\frac{\frac{1}{(z+1)\,\Gamma(z+1)}}{\frac{1}{z\,\Gamma(z)}} = \frac{e^{\gamma(z+1)} \prod_{n=1}^{\infty}\left(1 + \frac{z+1}{n}\right) e^{-\frac{z+1}{n}}}{e^{\gamma z} \prod_{n=1}^{\infty}\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}}$$

$$\frac{z\Gamma(z)}{(z+1)\Gamma(z+1)} = e^{\gamma} \prod_{n=1}^{\infty} \left(\frac{n+z+1}{n}\right)\left(\frac{n}{n+z}\right) e^{-\frac{1}{n}}$$

$$= e^{\gamma} \prod_{n=1}^{\infty} \frac{n+z+1}{n+z} e^{-\frac{1}{n}}$$

$$= e^{\gamma} \lim_{n\to\infty} \prod_{k=1}^{n} \frac{k+z+1}{k+z} e^{-\frac{1}{k}}$$

$$= e^{\gamma} \lim_{n\to\infty} \left(\frac{z+2}{z+1}\cdot\frac{z+3}{z+2}\cdots\frac{z+n+1}{z+n}\right) e^{-\sum_{n=1}^{\infty}\frac{1}{n}}$$

$$\left[\begin{array}{l} \gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - Log\, n\right) \Rightarrow e^{\gamma} = e^{\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - Log\, n\right)} \\ \Rightarrow e^{\gamma} = e^{\sum_{n=1}^{\infty}\frac{1}{n}} \cdot \lim_{n\to\infty}\frac{1}{n} \Rightarrow e^{-\sum_{n=1}^{\infty}\frac{1}{n}} = e^{-\gamma}\cdot \lim_{n\to\infty}\frac{1}{n} \end{array}\right]$$

$$= e^{\gamma} \lim_{n\to\infty} \frac{z+n+1}{z+1}\cdot\frac{1}{n}\cdot e^{-\gamma}$$

$$= \frac{1}{z+1}$$

Therefore,

$$\frac{z\Gamma(z)}{(z+1)\Gamma(z+1)} = \frac{1}{z+1} \Rightarrow \Gamma(z+1) = z\,\Gamma(z)\,.$$

$\boxdot$

Resources:

[1] Milton Abramowitz and Irene A. Stegun, Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, Tenth printing, 1972, p.255.

[2] Earl D. Rainville, Special Functions, The Macmillan Co., 1960.