Some notes on the gamma function

by Abdullah Aydemir, 12.01.2023

The classical gamma function in the integral form is defined by integrals

(1) $$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

(2) $$\Gamma(x) = \int_0^{\infty} t^{x} e^{-t} \frac{dt}{t}$$

(3) $$\Gamma(x) = \int_0^{1} (-logt)^{x-1} dt$$

(4) $$\Gamma(x) = \int_0^{1} \left(log\frac{1}{t}\right)^{x-1} dt$$

(5) $$\Gamma(x) = \int_0^{1} (-lnt)^{x-1} dt$$

(6) $$\Gamma(x) = \int_0^{1} \left(ln\frac{1}{t}\right)^{x-1} dt$$

(7) $$\Gamma(x) = \int_0^{1} (-lt)^{x-1} dt$$

(8) $$\Gamma(x) = \int_0^{1} \left(l\left(\frac{1}{t}\right)\right)^{x-1} dt$$

for non-zero and non-negative integers. On the other hand, the most popular definition of the gamma function is

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \qquad x > 0 \qquad \text{or} \qquad \Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \qquad Re(z) > 0.$$

Euler's limit based definition of the gamma function is

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)(x+2)\cdots(x+n)}, \qquad x \in \mathbb{R}, \qquad x \neq 0, -1, -2, \cdots$$

or

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)\cdots(z+n)}, \qquad z \in \mathbb{C}, \qquad z \neq 0, -1, -2, \cdots$$

The gamma function is defined by Leonhard Euler (1707-1783) to extend the values of $n!$ to non-integer real numbers. Euler considered x as the positive real numbers in the gamma function $\Gamma(x)$.

The notation Γ is due to Adrien-Marie Legendre (1752-1833). The gamma function is known as the Euler integral of the second kind. This naming is also due to Legendre.

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